

General Computer Science I (320101) Fall 2011

Practice Problem 1: SML Basics

10pt

Problem 1.1 (Head and Tail)

Program the following elementary list functions in SML and test them on three examples each. Please note that your program must work on every input except the empty list.

1. `head` takes a list L as input and returns the first element of L
2. `tail` takes a list L as input and returns the list consisting of all elements of L in original order except the first one.

Solution:

```
- fun head(h::t)=h;  
- fun tail(h::t)=t;
```

Problem 1.2: Define the `member` relation which checks whether an integer is member of a list of integers. The solution should be a function of type `int * int list -> bool`, which evaluates to `true` on arguments `n` and `l`, iff `n` is an element of the list `l`.

Solution: The simplest solution is the following

```
fun member(n,nil) = false
  | member(n,h::r) = if n=h then true else member(n,r);
```

The intuition here is that a is a member of a list l , iff it is the first element, or it is a member of the rest list.

Note that we cannot just use `member(n,n::r)` to eliminate the conditional, since SML does not allow duplicate variables in matching. But we can simplify the conditional after all: we can make use of SML's `orelse` function which acts as a logical “or” and get the slightly more elegant program

```
fun member(n,nil) = false
  | member(n,h::r) = (n=h) orelse member(n,r);
```

Problem 1.3: Define the `subset` relation. Set T is a subset of S iff all elements of T are also elements of S . The empty set is subset of any set.

Hint: Use the member function from Problem 1.2

Solution: Here we make use of SML's `andalso` operator, which acts as a logical "and"

```
fun subset(nil,_) = true
  | subset(x::xs,m) = member(x,m) andalso subset(xs,m);
```

The intuition here is that $S \subseteq T$, iff for some $s \in S$ we have $s \in T$ and $S \setminus \{s\} \subseteq T$.

Problem 1.4: Define functions to zip and unzip lists. zip will take two lists as input and create pairs of elements, one from each list, as follows: `zip [1,2,3] [0,2,4] ~> [[1,0],[2,2],[3,4]]`. unzip is the inverse function, taking one list of tuples as argument and outputting two separate lists. `unzip [[1,4],[2,5],[3,6]] ~> [1,2,3] [4,5,6]`.

Solution: Zipping is relatively simple, we will just define a recursive function by considering 4 cases:

```
fun zip nil nil = nil
  | zip nil l = l
  | zip l nil = l
  | zip (h::t) (k::l) = [h,k]::(zip t l)
```

Unzipping is slightly more difficult. We need map functions that select the first and second elements of a two-element list over the zipped list. Since the problem is somewhat under-specified by the example, we will put the rest of the longer list into the first list. To avoid problems with the empty tails for the shorter list, we use the `mapcan` function that appends the tail lists.

```
fun mapcan(f,nil) = nil | mapcan(f,h::t) = (f h)@(mapcan(f,t))
fun unzip (l) = if (l = nil) then nil
                else [(map head l),(mapcan tail l)]
```

Problem 1.5: Define a function `mymax` over SML lists of integers. Here, we take the maximum of an empty list to be the “default value” 0. You might want to consider a notation of type `mymax(x : y :: ys)` and work with the first two elements of the list in the step case.

Solution: In this program, we have to consider three cases,

- the empty list, there we give the default value
- the singleton list where the maximum is the unique value of the list
- lists of length 2 or more: Here the intuition for the recursion is that

$$\text{max}([x, y, \dots]) = \begin{cases} \text{max}([y, \dots]) & \text{if } x > y \\ \text{max}([x, \dots]) & \text{if } y \leq x \end{cases}$$

So we naturally get the following implementation:

```
fun mymax (nil) = 0
  | mymax ([x]) = x
  | mymax (x::y::xs) = if x > y then maxint(x::xs) else maxint (y::xs)
```

Note that with this recursion the first case is never reached, except if an empty list was given as the original argument.

Problem 1.6 (Factorial)

20pt

Write a recursive procedure `fact : int -> real`, that computes the factorial function $(n)! = 1 \cdot 2 \cdot \dots \cdot n$, where $(0)! = 1$. We want the procedure `fact` to diverge for negative arguments (on an abstract interpreter without resource limitations).

What is the largest number n you can compute $(n)!$ for on your system?

Solution:

```
- fun fact(0)=1.0| fact(n)=real(n)*fact(n-1)
- fact(170)
val it = 7.25741561531E306 : real
- fact(171);
val it = inf : real
```

The largest value is ∞ (infinity)! But seriously: the largest factorial that can be computed is $(170)!$.

Problem 1.7: Write three SML functions `nth`, `take`, `drop` that take a list and an integer as arguments, such that 6pt
20min

1. `nth(xs,n)` gives the `n`-th element of the list `xs`.
2. `take(xs,n)` returns the list of the first `n` elements of the list `xs`.
3. `drop(xs,n)` returns the list that is obtained from `xs` by deleting the first `n` elements.

In all cases, the functions should raise the exception `Subscript`, if `n` is negative or the list `xs` has less than `n` elements. We assume that list elements are numbered beginning with 0.

Solution:

```
exception Subscript
fun nth (nil,_) = raise Subscript
  | nth (h::t,n) = if n < 0 then raise Subscript
                  else if n=0 then h else nth(t,n-1)
fun take (l,0) = nil
  | take (nil,_) = raise Subscript
  | take (h::t,n) = if n < 0 then raise Subscript
                  else h::take(t,n-1)
fun drop (l,0) = l
  | drop (nil,_) = raise Subscript
  | drop (h::t,n) = if n < 0 then raise Subscript
                  else drop(t,n-1)
```

Problem 1.8: Write three SML functions `rev`, `take`, `drop` operating on lists (`lst`) and integers (`n`), such that 6pt
20min

1. `rev(lst)` returns the list `lst` in reversed order.
2. `last(lst,n)` returns the list of the last `n` elements of the list `lst`.

Make only use of the `::` and `@` built-in operators or those functions you have defined yourself. Note that `n` is always a positive integer. Assume that list elements are numbered beginning with 0.

Solution:

```
fun rev [] = []
  | rev (x::xs) = (rev xs) @ [x];
fun take (l,0) = nil
  | take (nil,_) = raise Subscript
  | take (h::t,n) = if n < 0 then raise Subscript
                    else h::take(t,n-1)
fun last (l,n) = rev(take(rev(l),n))
```

Problem 1.9: Write an SML function that tabulates lists, i.e `tabulate(f,n)` evaluates 10pt to the list `[f(0),\ldots,f(n-1)]`. As an example if `f(x)=2*x` and `n=[0,1,2]` then `tabulate(f,n)=[0,2,4]`

Problem 1.10: Declare a (abstract) data type for natural numbers and one for lists of natural numbers in SML. Write an SML function that given two natural number n and m (as a constructor term) creates the list $[n, n + 1, \dots, m - 1, m]$ if $n \leq m$ and raises an exception otherwise. 5pt
10min

Problem 1.11: Using the abstract data type of truth functions, give the defining equations for a function `myif` that takes three arguments, such that `myif(X,Y,Z)` behaves like “if `X` then `Y`, else `Z`”.

15pt

Hint: There is a control structure `if...then...else` in SML, of course you are not supposed to use that.

Problem 1.12: Write three variants of the `member` function in SML, where `member(x, xs)` 30pt returns `true`, iff `x` is an element in `xs`.

1. the first variant should not use another function.
2. the second variant should be non-recursive, using the function `myexists` (write that as well) that takes a property `p` and a list `l` as arguments and returns `true`, iff there is an element `a` in `l` such that `p(a)` evaluates to `true`.
3. the third variant should be non-recursive using the `foldl` function.

Solution:

```
fun member1 (x,nil) = false
  | member1 (x,h::t) = if (h=x) then true else member1(x,t)
fun myexists (_,nil) = false
  | myexists (p,h::t) = p(h) orelse myexists(p,t)
fun member2 (x,xs) = myexists (fn (y) => x=y) xs
fun member3 (x,xs) = foldl (fn (y,b) => b orelse x=y) false
```

Problem 1.13: Generalize the `mymax` function from Problem 1.5 to general lists with an ordering function (i.e. a function of type `('a * 'a) ->bool`) and a default value of type `'a`, which are passed as arguments. Given a list `l` an ordering relation `ord`, and a default value `d`, calling `gmax(l,ord,d)` evaluates to the `ord`-maximal element of `l`, or `d`, if `l` is empty.

Test this on lists of digits with the ordering relation given by $1 < 3 < 2 < 5 < 7 < 8 < 4 < 9 < 6$.

Solution: Generalizing the function is rather easy, we only add two arguments `ord` and `default` to the argument pattern of the function, and use them instead of the fixed values in `mymax`.

```
fun gmax (nil,_,default) = default
  | gmax ([x],_,_) = x
  | gmax (x::y::y::xs,ord,default) = if ord(x,y)
                                     then gmax(x::xs,ord,default)
                                     else gmax(y::xs,ord,default)
```

For the test, we need to come up with an predicate for the ordering relation given by $1 < 3 < 2 < 5 < 7 < 8 < 4 < 9 < 6$. We make use of the fact that SML looks at overlapping clauses in top-to-bottom order.

```
fun myord (1,_) = true | myord (_,1) = false
  | myord (3,_) = true | myord (_,3) = false
  | myord (2,_) = true | myord (_,2) = false
  | myord (5,_) = true | myord (_,5) = false
  | myord (7,_) = true | myord (_,7) = false
  | myord (8,_) = true | myord (_,8) = false
  | myord (4,_) = true | myord (_,4) = false
  | myord (9,_) = true | myord (_,9) = false
  | myord (6,_) = true | myord (_,6) = false
```

of course we get a warning that the match is non-exhaustive (we have not dealt with all integers) but for the moment we do not care.

Problem 1.14 (Infinite Precision Arithmetics)

We represent natural numbers of arbitrary length by non-empty lists of digits. Write SML functions for the arithmetical operations of addition, multiplication, integer division and modulo.

Problem 1.15 (Sorting a list)

Write a function that takes a list of strings and sorts it in ascending order like in dictionary.

Solution:

```
fun selmax(nil:string list) = nil
| selmax(a::nil) = a::nil
| selmax(a::b::t) =
    if a>b then b::selmax(a::t)
    else a::selmax(b::t);
fun descend(nil: string list) = nil
| descend(h::nil) = h::nil
| descend(l) = let
    val max = rev(selmax(l))
    in hd(max)::descend(tl(max))
end;
fun ascend(l) = rev(descend(l));
```

Problem 1.16: Write a function `split` that takes a string (a sentence) and returns a list of all pairs of strings one can get by splitting the sentence between any two words. A word is defined as a sequence of symbols between two spaces or a space and nothing. For example,

```
split "We_really_love_SML!";  
val it = [("We", "really_love_SML!"), ("We_really", "love_SML!"),  
("We_really_love", "SML!")]
```

Solution:

```
fun split(s) = let  
  fun go(sack, nil) = nil |  
    go(sack, #" " :: t1) = (implode(sack), implode(t1)) :: go(sack@["# "], t1) |  
    go(sack, w :: t1) = go(sack@[w], t1)  
  in  
    go([], explode(s))  
  end
```

Problem 1.17 (Ordering Function)

Write an SML function that takes an int list and sorts it in ascending order. For example:

```
sort [7,4,1,3];  
val it = [1,3,4,7];
```

Solution:

```
fun put(x,nil) = [x] |  
  put(x,h::t) = if (x<h) then x::h::t else h::put(x,t);  
fun sort(nil) = nil |  
  sort(h::t) = put(h,sort(t));
```

Problem 1.18: Write a recursive higher-order SML function `mapcan` that maps a list-valued function f over a list and appends all the result lists to a single list. What is the SML type of this function (explain).() 20pt

Write versions of `map` and `mapcan` that map a binary function over two lists. What are the SML types of these functions (explain).() 10 pt

Solution:

```
fun mapcan(f,nil) = nil | mapcan(f,h::t) = (f h)@(mapcan(f,t))
```

The SML type of `mapcan` is $(\text{'a} \rightarrow \text{'b list}) * \text{'a list} \rightarrow \text{'b list}$. The function `mapcan` takes two arguments: a list valued function, which has the type $(\text{'a} \rightarrow \text{'b list})$ and a list of some type 'a list . The output of the function is the concatenation of the resulting lists so it is again a list 'b list .

```
fun map2(f,nil,_) = nil
  | map2(f,_,nil) = nil
  | map2(f,h1::t1,h2::t2) = f(h1,h2)::map2(f,t1,t2)
```

`map2` has the type $(\text{'a} * \text{'b} \rightarrow \text{'c}) * \text{'a list} \rightarrow \text{'c list}$ because the function `map` takes two arguments: a binary function (type $\text{'a} * \text{'b} \rightarrow \text{'c}$) and a list (type 'a list). The result is again a list of some type 'c list .

```
fun mapcan2(f,nil,_) = nil
  | mapcan2(f,_,nil) = nil
  | mapcan2(f,h1::t1,h2::t2) = f(h1,h2)@mapcan2(f,t1,t2)
```

`mapcan2` has type $(\text{'a} * \text{'b} \rightarrow \text{'c}) * \text{'a list} \rightarrow \text{'c list}$ which is the same as above because the function `mapcan` takes the same arguments and returns a list. the difference is (not obvious from type) that the `mapcan` function does not return 'a list list type as `map` does if function f is a list valued function.

Note that in the above 'a , 'b , 'c define any SML type. Most of the above given functions work with all SML types, so a defined type like `int` for example is not required.

Problem 1.19: Write a non-recursive variant of the `member` function from Problem 1.2 using the `foldl` function.

Solution:

```
fun member (x,xs) = foldl (fn (y,b) => b orelse x=y) false
```

Problem 1.20 (List functions via foldl/foldr)

30pt

Write the following procedures using foldl or foldr

1. length which computes the length of a list
2. concat, which gets a list of lists and concatenates them to a list.
3. map, which maps a function over a list
4. myfilter, myexists, and myforall from ??

Solution:

```
fun length xs = foldl (fn (x,n) => n+1) 0 xs
fun concat xs = foldr op@ nil xs
fun map f = foldr (fn (x,yr) => (f x)::yr) nil
fun myfilter f =
  foldr (fn (x,ys) => if f x then x::ys else ys) nil
fun myexists f = foldl (fn (x,b) => b orelse f x) false
fun myall f = foldl (fn (x,b) => b andalso f x) true
```

Problem 1.21 (Understanding map)

6pt

Given the SML higher-order function `fun f x = fn (y) => y::x` and the list `val l = [1,2,3]`.

- Determine the types of `f` and `map (f l)`
- evaluate the expression `map (f l) l`

Solution:

```
- fun f x = fn (y) => y::x;
val f = fn : 'a list -> 'a -> 'a list
- map (f l);
val it = fn : int list -> int list list

- map (f l) l;
val it = [[1,1,2,3],[2,1,2,3],[3,1,2,3]] : int list list
```

Problem 1.22: Write a function `napply` that takes a function `f: int->int` and an integer `n`, and returns the result of n applications of f to itself, starting with number `n` as its argument. What does `napply(fn x=>x+1, n)` numerically compute? Is there any `n` for which `napply(fn x=>x div 2, n)` returns a non-zero value?

Solution: `fun napply (f, n) = let fun loop(0) = n — loop(a) = f(loop(a-1)) in loop(n) end`

Problem 1.23 (Nary Multiplication)

By defining a new datatype for n -tuples of unary natural numbers, implement an n -ary multiplications using the function `mul` from Problem 1.25. For $n = 1$, an n -tuple should be constructed by using a constructor named `first`; for $n > 1$, further elements should be prepended to the first by using a constructor named `next`. The multiplication function `nmul` should return the product of all elements of a given tuple.

For example,

```
nmul(next(s(s(zero)),
        next(s(s(zero))),
        first(s(s(s(zero))))))
```

should output `s(s(s(s(s(s(s(s(s(s(s(zero))))))))))` since $223 = 12$.

Solution:

```
datatype tuple = first of nat | next of nat*tuple;
fun nmult(first(num)) = num |
  nmult(next(num, rest)) = mult(num, nmult(rest));
```

Problem 1.24: Declare a data type `myNat` for unary natural numbers and `NatList` for lists of natural numbers in SML syntax, and define a function that computes the length of a list (as a unary natural number in `myNat`). Furthermore, define a function `nms` that takes two unary natural numbers `n` and `m` and generates a list of length `n` which contains only `ms`, i.e. `nms(s(s(zero)),s(zero))` evaluates to `construct(s(zero),construct(s(zero),elist))`. 4pt.
8min

Solution:

```
datatype mynat = zero | s of mynat;  
datatype natlist = elist | construct of mynat * natlist;  
fun length (nil) = zero | length (construct (n,l)) = s(length(l));  
fun nms(zero,m) = elist | nms(s(n),m) = construct(m,nms(n));
```

Problem 1.25 (Unary natural numbers)

Define a datatype `nat` of unary natural numbers and implement the functions

- `add = fn : nat * nat -> nat` (adds two numbers)
- `mul = fn : nat * nat -> nat` (multiplies two numbers)

Solution:

```
datatype nat = zero | s of nat;  
fun add(zero:nat,n2:nat) = n2  
  | add(n1,zero) = n1  
  | add(s(n1),s(n2)) = s(add(n1,s(n2)));  
fun mult(zero:nat,_) = zero  
  | mult(_,zero) = zero  
  | mult(n1,s(zero)) = n1  
  | mult(s(zero),n2) = n2  
  | mult(n1,s(n2)) = add(n1,mult(n1,n2));
```

Problem 1.26 (Binary and Decimal Addition)

Define binary and decimal addition, and multiplication on lists of digits.

Hint: This is just a sneaky way of getting you to practice with lists.

For the decimal addition function, we represent natural numbers as lists of of digits, e.g. `[5,3,4]` for the number 534. Now the function `badd` works as follows: `badd([2,8],[1,3])` evaluates to `[4,1]`.

The binary addition function is similar, only have it operates on lists of binary digits, i.e. the numbers 1 and 0.

Problem 1.27: Program the function f with $f(x) = x^2$ on unary natural numbers 15pt
without using the multiplication function.

Solution: We will use the abstract data type `mynat`

```
datatype mynat = zero | s of mynat
fun add(n,zero) = n | add(n,s(m))=s(add(n,m))
fun sq(zero)=zero|sq(s(n))=s(add(add(sq(n),n),n))
```

Problem 1.28 (Floating Point Powers)

15pt

Write a recursive procedure `power : real * int -> real`, that computes the power x^n for a real number x and a natural number n by floating point operations. What is the result of `power(3.0, 100)`, is this really the number 3^{100} (discuss)?

Solution:

```
- fun power(x:real,0) = 1.0 | power(x,n) = x * power(x,n-1);  
- power(3.0,100)  
val it = 5.15377520732E47 : real
```

This is not the number 3^{100} , since it is a floating point approximation; The result has 36 zeros at the end, but 3^{100} does not.

Problem 1.29 (Your own lists)

Define a data type `mylist` of lists of integers with constructors `mycons` and `mynil`. Write translators `tosml` and `tomy` to and from SML lists, respectively.

Solution: The data type declaration is very simple

```
datatype mylist = mynil | mycons of int * mylist;
```

it declares three symbols: the base type `mylist`, the individual constructor `mynil`, and the constructor function `mycons`.

The translator function `tosml` takes a term of type `mylist` and gives back the corresponding SML list; the translator function `tomy` does the opposite.

```
fun tosml mynil = nil
  | tosml mycons(n,l) = n::tosml(l)
fun tomy nil = mynil
  | tomy (h::t) = mycons(h,tomy(t))
```

Problem 1.30: Write a function that takes an odd positive integer and returns a char list which represents a triangle of stars with n stars in the last row. For example,

```
triangle 5;
val it =
[#"  ", #"  ", #"*", #"  ", #"  "],
[#"  ", #"*", #"*", #"*", #"  "],
[#"*", #"*", #"*", #"*", #"*"]
```

Solution:

```
fun stars(0) = nil |
  stars(n) = #"*" :: stars(n-1)

fun wall(nil) = nil |
  wall(hd::tl) = ((#" " :: hd)@[" "])::wall(tl)

fun triangle(1) = [[#"*"]] |
  triangle(n) = wall(triangle(n-2))@[stars(n)];
```

Problem 1.31 (Generating Function)

Write an SML function of type `combine: int -> int list list` that takes an integer n and returns a list of 2^n lists of length n which represent all combinations of ones and zeros (as in a truth table).

For example:

```
combine 2;  
val it = [[0,0],[0,1],[1,0],[1,1]] : int list list
```

Solution:

```
fun combine 0 = [[]] |  
  combine x = let  
    val y = combine(x-1)  
    fun app (v, nil ) = nil |  
          app (v, hd::tl) = (v::hd)::app(v, tl);  
  in  
    app(0, y) @ app(1, y)  
  end;
```

Problem 1.32 (Permutations)

20pt

1. Implement a function with the following type

```
intersperse: 'a -> 'a list -> 'a list list
```

The function takes a List `xs`, an element `y` and calculates all available lists, whereas `y` has been inserted into `xs` at an arbitrary position. Example:

```
intersperse 1 [2,3] = [[1,2,3],[2,1,3],[2,3,1]]
```

2. Implement a function with the following type

```
permutations: 'a list -> 'a list list
```

that gives you all permutations of a list. Example:

```
permutations [1,2,3] = [[1,2,3],[2,1,3],[2,3,1],[1,3,2],[3,1,2],[3,2,1]]
```

Whether or not you successfully implemented the function `intersperse`, you may use it here.

Solution:

```
fun intersperse a [] = [[a]]
  | intersperse a (x::xs) = (a::x::xs) :: (map (fn z => (x::z)) (intersperse a xs));

fun permutations [] = [[]]
  | permutations (x::xs) =
    let
      fun mymap (h::tl) = (intersperse x h) @ mymap tl
        | mymap nil = nil
    in mymap (permutations xs) end;
```
